

ON THE GROWTH OF LOCAL INTERSECTION MULTIPLICITIES IN HOLOMORPHIC DYNAMICS: A CONJECTURE OF ARNOLD

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ABSTRACT. We show by example that local intersection multiplicities in holomorphic dynamical systems can grow arbitrarily fast, answering a question of V. I. Arnold.

INTRODUCTION

Let $f: (\mathbf{C}^2, 0) \rightarrow (\mathbf{C}^2, 0)$ be a germ of a holomorphic map that fixes the origin $0 \in \mathbf{C}^2$, and which is finite-to-one near 0. Suppose C and D are two germs of holomorphic curves passing through 0. In this article, we will study the sequence of local intersection multiplicities $\mu(n) := C \cdot f^n(D)$ at the origin, for $n \geq 0$. Specifically, we will address the question: *how fast can the sequence $\mu(n)$ grow?* This and related questions have been posed and studied by V. I. Arnold, who conjectured that, if $\mu(n) < \infty$ for every n , the sequence $\mu(n)$ grows at most exponentially fast, see [Arn93], [Arn94, p. 215] and [Arn04, problems 1994-49 and 1994-50]. Arnold proved the conjecture in the case when f is a local biholomorphism and in some cases when the complex derivative $f'(0)$ has exactly one zero eigenvalue [Arn93], but the general case appears to be unknown.

In this article, we will see by explicit construction that Arnold's conjecture is false in general, and that in fact the sequence $\mu(n)$ can grow arbitrarily fast. More precisely, we will prove the following theorem.

Main Theorem. *Let $f: \mathbf{C}^2 \rightarrow \mathbf{C}^2$ be the polynomial map $f(x, y) = (x^2 - y^4, y^4)$, and let $\nu: \mathbf{N} \rightarrow \mathbf{R}$ be any function. Then there exist germs of holomorphic curves C and D through the origin such that the local intersection multiplicities $\mu(n) = C \cdot f^n(D)$ are always finite, and such that $\mu(n) > \nu(n)$ for infinitely many n .*

Notice that the complex derivative $f'(0)$ for this map is 0, so f defines a *superattracting* germ at 0. The dynamics of superattracting germs is an active area of research in holomorphic dynamics in several variables, see for instance [HP94, Fav00, FJ07, Rug12, Rug11, CAR11, BEK11, GR12] and the notes [Jon12].

Before beginning the proof of the theorem in earnest, we begin with a brief overview. Let S be the space of binary sequences $S = \{0, 1\}^{\mathbf{N}}$, and let $\sigma: S \rightarrow S$ denote the left-shift map on S . For any two sequences $s, t \in S$, set $M(s, t)$ to be the smallest index m such that $s_m \neq t_m$, with $M(s, t) = \infty$ if $s = t$. To prove the theorem, we will construct a family $\{C_s\}_{s \in S}$ of germs of holomorphic curves through the origin with the properties that

1. $f(C_s) = C_{\sigma(s)}$ for each $s \in S$, and
2. for any $s, t \in S$, the local intersection multiplicity $C_s \cdot C_t$ is $\asymp 4^{M(s, t)}$.

The theorem then follows easily from the following simple proposition, the proof of which is left to the reader.

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Proposition. *Let $\nu: \mathbf{N} \rightarrow \mathbf{R}$ be any function. Then there exist sequences $s, t \in S$ such that $M(s, \sigma^n(t))$ is finite for all $n \geq 0$, and such that $M(s, \sigma^n(t)) > \nu(n)$ for infinitely many n .*

In §1, we will construct the C_s as *formal curves*, that is, as curves defined by irreducible formal power series $\varphi_s \in \mathbf{C}[[x, y]]$. The coefficients of the power series φ_s will be determined via a recursive procedure that guarantees properties 1. and 2. are satisfied. In §2, we will prove that each formal power series φ_s is actually convergent, and hence that the curve germs C_s are *holomorphic*. It should be noted that the construction of the power series φ_s in §1 is purely algebraic, and that if we replace the word *holomorphic* with *formal*, the theorem holds when \mathbf{C} is replaced by any field of characteristic $\neq 2$. Finally, it is worth mentioning that this counterexample is by no means isolated; one can construct similar “Cantor sets of curves” for many other superattracting germs.

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1. THE CONSTRUCTION

For the rest of the article, we let $f: \mathbf{C}^2 \rightarrow \mathbf{C}^2$ be the polynomial map $f(x, y) = (x^2 - y^4, y^4)$, S be the space of binary sequences $S = \{0, 1\}^{\mathbf{N}}$, and $\sigma: S \rightarrow S$ be the left-shift map on S . We now define a family $\{\varphi_s\}_{s \in S}$ of irreducible formal power series $\varphi_s \in \mathbf{C}[[x, y]]$ of the form

$$(1.1) \quad \varphi_s(x, y) = x + a_0^s y^2 + a_1^s y^6 + \cdots + a_n^s y^{2+4n} + \cdots,$$

by recursively defining the coefficients a_n^s in the following way. First, set $a_0^s = (-1)^{s_0}$; then, assuming a_0^s, \dots, a_n^s have been defined for all $s \in S$, we set

$$(1.2) \quad a_{n+1}^s = \begin{cases} -\frac{1}{2a_0^s} \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} a_i^s a_j^s & \text{if } 4 \nmid n. \\ -\frac{a_{n/4}^{\sigma(s)}}{2a_0^s} - \frac{1}{2a_0^s} \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} a_i^s a_j^s & \text{if } 4 \mid n. \end{cases}$$

Let C_s denote the formal curve through the origin in \mathbf{C}^2 defined by φ_s . We now spend the rest of the section proving the following proposition.

Proposition 1.1. *The formal curves $\{C_s\}_{s \in S}$ satisfy*

1. $f(C_s) = C_{\sigma(s)}$ for all $s \in S$, and
2. the local intersection multiplicity $C_s \cdot C_t$ is $\frac{1}{3}(4^{m+1} + 2)$, where m is the smallest index such that $s_m \neq t_m$.

Proof. To prove (1), we must show that $\varphi_s \mid (\varphi_{\sigma(s)} \circ f)$ in the ring $\mathbf{C}[[x, y]]$. Indeed, we will show that $\varphi_{\sigma(s)} \circ f = (x + a_0^s y^2 + a_1^s y^6 + \cdots)(x - a_0^s y^2 - a_1^s y^6 - \cdots)$. To see this, first observe that

$$(1.3) \quad (x + a_0^s y^2 + a_1^s y^6 + \cdots)(x - a_0^s y^2 - a_1^s y^6 - \cdots) = x^2 - y^4 - \sum_{n \geq 0} \sum_{\substack{i+j=n+1 \\ i,j \geq 0}} a_i^s a_j^s y^{4(n+2)}.$$

The recursion formula (1.2) gives that the coefficient of $y^{4(n+2)}$ in this expression is 0 when $4 \nmid n$ and is $a_{n/4}^{\sigma(s)}$ when $4 \mid n$, so the right hand side of (1.3) is

$$x^2 - y^4 + \sum_{k \geq 0} a_k^{\sigma(s)} y^{4(4k+2)} = x^2 - y^4 + \sum_{k \geq 0} a_k^{\sigma(s)} y^{8+16k} = \varphi_{\sigma(s)} \circ f.$$

This completes the proof of (1).

To prove (2), we first make the easy observation that the intersection multiplicity $C_s \cdot C_t$ is precisely the smallest integer k such that the coefficients of y^k in the power series φ_s and φ_t are different. From equation (1.1), it then follows that $C_s \cdot C_t = 2 + 4n$, where n is the smallest index such that $a_n^s \neq a_n^t$. We will prove (2) by induction on $m \geq 0$, where m is the smallest index such that $s_m \neq t_m$. If $m = 0$, then $a_0^s \neq a_0^t$, and hence $C_s \cdot C_t = 2$, establishing the base case of the induction. Now assume that $m > 0$ is the smallest index such that $s_m \neq t_m$. Then, by induction, $C_{\sigma(s)} \cdot C_{\sigma(t)} = \frac{1}{3}(4^m + 2)$, from which it follows that the first index n for which $a_n^{\sigma(s)} \neq a_n^{\sigma(t)}$ is $n = \frac{1}{3}(4^{m-1} - 1)$. Using the recursion formula (1.2), we can then conclude that the first index n such that $a_n^s \neq a_n^t$ is

$$n = 1 + \frac{4}{3}(4^{m-1} - 1) = \frac{1}{3}(4^m - 1).$$

Thus $C_s \cdot C_t = 2 + \frac{4}{3}(4^m - 1) = \frac{1}{3}(4^{m+1} + 2)$, completing the induction, and the proof. \square

2. ANALYTICITY

In this section, we will prove using the recursion formula (1.2) that each of the power series φ_s constructed in §1 are convergent. Indeed, using very crude estimates, we will prove the following proposition.

Proposition 2.1. *Let $C = 1/20$ and $R = 10$. Then $|a_n^s| \leq CR^n/n^2$ for each $n \geq 1$ and each $s \in S$. In particular, φ_s converges on the set $\{(x, y) \in \mathbf{C}^2 : |y| < 1/10\}$.*

To prove the proposition, we will make use of the following lemma.

Lemma 2.2. *Let $n \geq 1$ be an integer. Then*

$$\sum_{k=1}^n \frac{1}{k^2(n-k+1)^2} \leq \frac{20}{(n+1)^2}.$$

Proof. The symmetry in the terms of the left hand sum implies that

$$\sum_{k=1}^n \frac{1}{k^2(n-k+1)^2} \leq 2 \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{k^2(n-k+1)^2}.$$

Multiplying both sides of this inequality by $(n+1)^2$ yields

$$\sum_{k=1}^n \frac{(n+1)^2}{k^2(n-k+1)^2} \leq 2 \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(n+1)^2}{k^2(n-k+1)^2} = 2 \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{k^2(1 - \frac{k}{n+1})^2} \leq 8 \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{k^2} < \frac{8\pi^2}{6}.$$

Since $8\pi^2/6 < 20$, the proof is complete. \square

Proof of Proposition 2.1. We will prove the proposition by induction on $n \geq 1$. When $n = 1$, the recursion formula (1.2) gives $a_1^s = -a_0^{\sigma(s)}/2a_0^s = \pm \frac{1}{2}$ for each $s \in S$, and hence $|a_1^s| = \frac{1}{2} = CR$, establishing the base case of the induction. Now assume that the proposition holds for a_k^s when $k \leq n$. If $4 \nmid n$, then (1.2), the triangle inequality, and Lemma 2.2 give that

$$|a_{n+1}^s| \leq \frac{1}{2} \sum_{k=1}^n \frac{C^2 R^{n+1}}{k^2(n-k+1)^2} \leq \frac{20C^2 R^{n+1}}{2(n+1)^2} = \frac{CR^{n+1}}{2(n+1)^2} < \frac{CR^{n+1}}{(n+1)^2},$$

establishing the proposition in this case. If $4 \mid n$, then (1.2) gives

$$(2.1) \quad |a_{n+1}^s| \leq \frac{CR^{n/4}}{2(n/4)^2} + \frac{1}{2} \sum_{k=1}^n \frac{C^2 R^{n+1}}{k^2(n-k+1)^2} \leq \frac{CR^{n/4}}{2(n/4)^2} + \frac{CR^{n+1}}{2(n+1)^2}.$$

Since $4 \mid n$, and in particular $n \geq 4$, the inequality $\frac{n}{4} \leq (n+1) - 4$ is valid, and hence

$$\frac{CR^{n/4}}{2(n/4)^2} = \frac{8CR^{n/4}}{n^2} \leq \frac{8CR^{n+1}}{n^2 R^4} < \frac{8CR^{n+1}}{(n+1)^2 R^4}.$$

Putting this estimate into (2.1), we see that

$$|a_{n+1}^s| \leq \left(\frac{8}{R^4} + \frac{1}{2} \right) \frac{CR^{n+1}}{(n+1)^2} < \frac{CR^{n+1}}{(n+1)^2}.$$

This completes the proof. \square

REFERENCES

- [Arn93] Vladimir Arnold. Bounds for Milnor numbers of intersections in holomorphic dynamical systems. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 379–390. Publish or Perish, Houston, TX, 1993.
- [Arn94] Vladimir Arnold. Sur quelques problèmes de la théorie des systèmes dynamiques. *Topol. Methods Nonlinear Anal.*, 4(2):209–225, 1994.
- [Arn04] Vladimir Arnold. *Arnold's problems*. Springer-Verlag, Berlin, 2004. Translated and revised edition of the 2000 Russian original, With a preface by V. Philippov, A. Yakivchik and M. Peters.
- [BEK11] Xavier Buff, Adam Epstein, and Sarah Koch. Böttcher coordinates. 2011. To appear in *Indiana Univ. Math. J.*
- [CAR11] Eduardo Casas-Alvero and Joaquim Roé. Iterated inverse images of plane curve singularities. *Acta Math. Sin. (Engl. Ser.)*, 27(12):2319–2328, 2011.
- [Fav00] Charles Favre. Classification of 2-dimensional contracting rigid germs and Kato surfaces. I. *J. Math. Pures Appl. (9)*, 79(5):475–514, 2000.
- [FJ07] Charles Favre and Mattias Jonsson. Eigenvaluations. *Ann. Sci. École Norm. Sup. (4)*, 40(2):309–349, 2007.
- [GR12] William Gignac and Matteo Ruggiero. Growth of attraction rates for iterates of a superattracting germ in dimension two. 2012. arXiv:1209.3450.
- [HP94] John Hubbard and Peter Papadopol. Superattractive fixed points in \mathbf{C}^n . *Indiana Univ. Math. J.*, 43(1):321–365, 1994.
- [Jon12] Mattias Jonsson. Dynamics on Berkovich spaces in low dimensions. 2012. arXiv:1201.1944.
- [Rug11] Matteo Ruggiero. Contracting rigid germs in higher dimensions. 2011. To appear in *Ann. Inst. Fourier (Grenoble)*.
- [Rug12] Matteo Ruggiero. Rigidification of holomorphic germs with noninvertible differential. *Michigan Math. J.*, 61(1):161–185, 2012.

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